SOME TOPOLOGICAL PROPERTIES OF HALFGROUPOIDS[1]

by

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This thesis is primarily an investigation of the structure of a very general algebraic system, the halfgroupoid, in relation to a particular topology, the (left) ideal topology. An extension of this study to the theory of graphs is discussed in detail, while some other applications and related problems are mentioned only briefly.

If H is a halfgroupoid with binary operation $\circ$, and I_L is a subset, possibly empty, of H such that $H \circ I_L \subseteq I_L$, then I_L is a left ideal for H, and the set of all such left ideals on a given halfgroupoid is proven to constitute a topology with completely additive closure. The mapping of the family of all halfgroupoids into the family of all possible (left) ideal topologies is found to be a many-to-one correspondence, and examples of different halfgroupoids having the same topology are given. Although the halfgroupoid operation need not be continuous under the (left) ideal topology, a sufficient condition for continuity is presented. Also, both necessary and sufficient conditions for the topological separation axioms T_0 and T_1 to be satisfied are proven, and the property of topological connectedness is investigated specifically in terms of (left) ideals.

Extending these results, mappings between the family of all halfgroupoids and the family of all directed graphs are constructed in such a way as to establish a topological correspondence between these two mathematical systems, whereby the (left) ideal topology is found to be identical to the previously known digraph topology. A short synopsis is given of research done elsewhere on the problem of relating algebra and topology from the converse point of view, and brief mention is made of various other possibilities for extending or applying the results in this thesis to other fields of mathematics. Finally, in the Appendix there is a short outline of some other ways of establishing connections among algebra, topology, and graphs to yield useful results appropriate to various situations.

PREFACE

In this thesis our primary purpose is to investigate the relationship between a particular topological space and a very general algebraic structure. We also discuss some applications and extensions of our results and the relevance of our findings to some previously known results.

With sincere appreciation, we gratefully thank Dr. T. N. Bhargava for his inspiring encouragement and patient guidance throughout the development of this thesis. We are also deeply indebted to Dr. P. H. Doyle (Mighigan State University, formerly at Virginia Polytechnic Institute) for his many thoughtful comments and helpful suggestions regarding this research.

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TABLE OF CONTENTS

INTRODUCTION

One popular and practical trend in mathematical research is that toward attempting to discover fundamental similarities between the many diverse and increasingly specialized fields of mathematics. Generally it is hoped that through these efforts certain basic principles will be found to be present and consistent throughout various subjects in mathematics, thereby tending to increase the unity and general understanding of all mathematical ideas.

In this thesis we follow this trend by investigating some relationships between the most general algebraic system - the halfgroupoid, which consists of a set H of elements and a binary operation - and a specific topological system - the (left) ideal topology, in which the open sets are subsets $I \subseteq H$ such that $H \circ I \subseteq I$. Our approach is primarily that of interpreting certain topological properties in terms of the structure of the halfgroupoid.

In Chapter One we explain the general notation to be used throughout our discussion and present some preliminary definitions of some basic algebraic concepts. In Chapter Two we establish our (left) ideal topology on an arbitrary half-groupoid and then study such things as continuity of operation in the halfgroupoid, the topological separation axioms, and topological connectedness.

In Chapter Three we extend our analysis of algebra and topology to include a third field of mathematics, the theory of graphs. We show that because a halfgroupoid may be thought of as a set on which is defined a ternary relation, while a directed graph is a set on which is defined a binary relation, these two mathematical systems are somewhat naturally related, and we can, in fact, formulate a topological correspondence between them by defining a mapping between halfgroupoids and digraphs under which our (left) ideal topology proves to be identical to the digraph topology previously developed by Ahlborn (see [1]). Thus we create a three-way connection between algebra, topology, and graph theory. Next, we discuss briefly a problem utilizing an approach which is basically converse to ours, and finally we mention various ways in which the results presented in this thesis may be applied or extended to other fields of mathematics.

CHAPTER ONE

PRELIMINARIES

We begin by explaining in general terms the notation to be used throughout this thesis. In the second section we present some background material, primarily in the form of various standard definitions.

1.1 NOTATION

Throughout our discussion we use fairly standardized notation, especially as found in Doyle and Warne [8], Kelley [10], and Bhargava [3]. For example, sets are denoted by capital letters A, B, C,..., and families of sets by large script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$,... . Individual elements of sets are denoted by small letters a, b, c,..., which may be underlined to avoid confusion with other words; a statement to the effect that a belongs to the set A is often shortened to $a \in A$. The empty set is denoted by ϕ, and the complement of the set S, by $\tilde{S}$. A distinction is made between a general subset $A \subseteq B$, and a proper subset $A \subset B$.

The set operations of union, intersection, and Cartesian product are written $\cup$, $\cap$, and $\times$, respectively, and a notation such as $\bigcap\{S_p\}$ indicates that the intersection is taken of all sets S_p having the particular property under consideration, whereas $\bigcap_\alpha\{S_\alpha\}$ indicates that the intersection of the sets S_α is taken over an arbitrary index set, of which α is a general element.

Other, more specific, notation will be explained as the need arises, as in the statements of definitions or theorems.

1.2 DEFINITIONS

In this section we define some of the basic concepts used in algebra, particularly in connection with halfgroupoids (see Doyle and Warne [8] or Bruck [7]. Most of these terms are ones to which we repeatedly refer throughout our discussion; a few others are included for their general interest. Terminology related to other fields, such as topology and graph theory, will be introduced when pertinent to the discussion of a particular topic.

Definition 1.2.1 A *mathematical system* is a set of elements and at least one operation defined for the set. (This operation may have certain properties and may be subject to various rules.) The *order* of a system is the cardinality of the set of elements in the system.

We note that because a mathematical system is thus identified with its set of elements, there will be little chance for confusion if we use the same capital letter to denote either the system or the set of elements comprising it.

Definition 1.2.2 A binary operation, denoted by $\circ$, and defined for a set S, is a single-valued mapping of a set $D \subseteq (S \times S)$, where $(S \times S) = \{(a,b): a,b \in S\}$, onto a set $R \subseteq S$. The set D is called the domain of definition of the operation, and the set R is called the range of image values. R is the empty set if, and only if, D is also empty; in such a case, we have a null system.

If $(a,b) \in D$, we say $a \circ b$ is defined, and if the image in R of (a,b) is c, we write $a \circ b = c$. If $(a,b) \in \tilde{D}$, we say $a \circ b$ is undefined.

If M and N are subsets of S, then $M \circ N$ is the subset of R consisting of the images of the elements contained in $(M \times N) \cap D$; that is, $M \circ N = \{c: a \circ b = c,\ a \in M,\ b \in N\}$.

The binary operation in a system of finite order is often completely exhibited in a Cayley table; we will have occasion to use this method in several examples to be given later.

It is very important to note here that, for our purposes, it is understood that a mathematical system has a certain property if this property is present wherever the operation is defined. Thus it may be assumed that, whenever the operation is undefined, any requirement is satisfied vacuously.

Definition 1.2.3 A halfgroupoid H is a mathematical system consisting of a nonempty set of elements for which is defined a binary operation having no special properties.

Definition 1.2.4 A groupoid G is a halfgoupoid H in which the domain of definition of the operation is the entire Cartesian product $H \times H$.

The remaining definitions in this section are formulated in terms of halfgroupoids, but of course apply in particular to groupoids.

Definition 1.2.5 A subhalfgroupoid S of a halfgroupoid H is a subsystem consisting of a nonempty subset of elements of H, such that $S \circ S \subseteq S$.

Definition 1.2.6 An antihalfgroupoid A of a halfgroupoid H is a subsystem consisting of a nonempty subset of elements of H, such that $A \circ A \subseteq \tilde{A}$.

We note that when the preceding two terms are applied to groupoids rather than halfgroupoids, the infix "half" is deleted so that the terms become subgroupoid and antigroupoid.

Definition 1.2.7 A left ideal for a halfgroupoid H is a subset $I_L \subseteq H$ such that $H \circ I_L \subseteq I_L$. Similarly, a right ideal is a subset $I_R \subseteq H$ such that $I_R \circ H \subseteq I_R$. A two-sided ideal, as the name implies, is a subset $I \subseteq H$ which is both a left and right ideal. A point ideal is any ideal consisting of a single element.

Here we make a special note of the fact that we will consider the empty set ϕ to be a trivial ideal, either left, right, or two-sided.

Because our discussion throughout this thesis is limited in general to consideration only of left ideals, the definitions below are stated so as to correspond to the left ideal, i.e. in terms of the right side. Extensions of these definitions to the left side, or both sides, can be made in an obvious manner.

Definition 1.2.8 If $a, b \in H$, then a is a right factor of b and b is a left multiple of a, in notation a/b, if there exists an element $c \in H$ such that $c \circ a = b$.

Definition 1.2.9 An element $a \in H$ is prime if the set of (right) factors of a is the empty set or $\{a\}$ itself. An element which is not prime is called composite. An element $a \in H$ for which $a \circ a = a$ is called an idempotent element.

Definition 1.2.10 An element $a \in H$ is a right zero element if for all $x \in H$, $x \circ a = a$.

Definition 1.2.11 An element $a \in H$ is a right unit element if for all $x \in H$, $x \circ a = x$.

CHAPTER TWO

RESULTS

In this chapter we establish a point-set topology on an arbitrary halfgroupoid and then investigate the relationships between some basic topological properties and the structure of the halfgroupoid. We refer to Kelley [10] for definitions of terms related to topology.

2.1 THE IDEAL TOPOLOGY

Definition 2.1.1 A topology defined on a set S is a mathematical system consisting of a family $\mathcal{T}$ of subsets of S, with the operations of union and intersection, such that the following are true:

i) S and ϕ are members of $\mathcal{T}$;

ii) the union of any number of members of $\mathcal{T}$ itself belongs to $\mathcal{T}$;

iii) the intersection of any two, and hence any finite number of, members of $\mathcal{T}$ itself belongs to $\mathcal{T}$.

The set S, along with a topology $\mathcal{T}$ on S, constitutes a topological space (S, $\mathcal{T}$). The members of $\mathcal{T}$ are called open sets in S and their complements are closed sets. An (open) neighborhood O_x of a point $x \in S$ is any (open) set containing $\underline{x}$.

A family $\mathcal{B}$ of subsets of S is a <u>base</u> for the topology $\mathcal{T}$ if the family of all possible unions of members of $\mathcal{B}$ is the family $\mathcal{T}$.

A family $\mathcal{S}$ of subsets of S is a <u>subbase</u> for the topology $\mathcal{T}$ if the family of all finite intersections of members of $\mathcal{S}$ yields a base $\mathcal{B}$ for $\mathcal{T}$.

<u>Definition 2.1.2</u> A topology $\mathcal{T}$ is said to have the property of <u>completely additive closure</u> if the intersection of <u>any</u> number of members of $\mathcal{T}$ is itself a member of $\mathcal{T}$.

<u>Theorem 2.1.1</u> The family $\mathcal{I}_L$ of all left ideals for a half-groupoid H constitutes a topology with the property of completely additive closure.

<u>Proof</u>: i) Clearly, $H \in \mathcal{I}_L$ because $H \circ H \subseteq H$, and $\phi \in \mathcal{I}_L$, by definition.

ii) Let $\{I_{L\alpha}\}$, where α is the general element of an arbitrary index set, be a set of elements of $\mathcal{I}_L$. Then, $\bigcup_\alpha\{I_{L\alpha}\} \in \mathcal{I}_L$, for if $x \in \bigcup_\alpha\{I_{L\alpha}\}$, then $x \in I_{Lk}$, for some fixed $\underline{k}$, and $y \circ x \in I_{Lk}$, for all $y \in H$. Hence, $y \circ x \in \bigcup_\alpha\{I_{L\alpha}\}$ for all $y \in H$ and, because $\underline{x}$ is arbitrary, for all $x \in \bigcup_\alpha\{I_{L\alpha}\}$. That is, $H \circ \bigcup_\alpha\{I_{L\alpha}\} \subseteq \bigcup_\alpha\{I_{L\alpha}\}$, and $\bigcup_\alpha\{I_{L\alpha}\} \in \mathcal{I}_L$.

iii) Also, $\bigcap_\alpha\{I_{L\alpha}\} \in \mathcal{I}_L$, for if $x \in \bigcap_\alpha\{I_{L\alpha}\}$, then $\underline{x}$ belongs to every $I_{L\alpha}$, and $y \circ x$ must thus belong to every $I_{L\alpha}$, for all $y \in H$. Hence $y \circ x \in \bigcap_\alpha\{I_{L\alpha}\}$ for all $y \in H$ and, again because $\underline{x}$ is arbitrary, for all $x \in \bigcap_\alpha\{I_{L\alpha}\}$. That is,

$H \circ \bigcap_{\alpha}\{I_{L\alpha}\} \subseteq \bigcap_{\alpha}\{I_{L\alpha}\}$, and $\bigcap_{\alpha}\{I_{L\alpha}\} \in \mathcal{I}_L$.

It is a well-known result that, in a topology $\mathcal{T}$ having completely additive closure, the family $\mathcal{C}$ of closed sets under $\mathcal{T}$ satisfies the same axioms as the family $\mathcal{O}$ of open sets. Thus one can obtain from $\mathcal{T}$ the dual topology $\mathcal{T}^*$ by considering the family $\mathcal{C}$ to be open sets. Hence it is actually immaterial in Theorem 2.1.1 whether we designate the family of left ideals to be the open or closed sets; in either case they define a topology. However, in obtaining various results relating halfgroupoids and topology it is more convenient and perhaps more natural to consider the ideals to be open sets. Therefore, we make the following formal definition:

Definition 2.1.3 The (left) ideal topology on a halfgroupoid H is the family of all (left) ideals for H.

Since throughout our discussion we do consider only left ideals, for simplicity we represent by $(H,\mathcal{I})$ the space of the (left) ideal topology on a halfgroupoid. This enables us to use the convenient notation I_x to denote a left ideal containing the point x (see Definition 2.1.1).

We should point out here though that, from the proof of Theorem 2.1.1, it is obvious that the set of all right ideals in H would also constitute a topology, as would the set of all two-sided ideals. However, the set of all possible right, left, and two-sided ideals in general is not closed under finite intersections and must therefore be used as a subbase to obtain a topology for H, as shown by the following example:

Example 2.1.1 Let the halfgroupoid H be defined as in this Cayley table:

	a	b	c
a	a	c	a
b	c	-	c
c	c	b	c

Then $\mathcal{I}_L = \{\phi, H, \{a,c\}\}$ and $\mathcal{I}_R = \{\phi, H, \{b,c\}\}$, so the family $\mathcal{I}_L \cup \mathcal{I}_R = \{\phi, H, \{a,c\}, \{b,c\}\}$. But $\{a,c\} \cap \{b,c\} = \{c\}$, which is neither a right nor a left ideal for H. However, letting $\mathcal{I}_L \cup \mathcal{I}_R$ be a subbase, we obtain $\mathcal{T} = \{\phi, H, \{a,c\}, \{b,c\}, \{c\}\}$, which is indeed a topology on H.

It is also interesting to note here that, whereas a given halfgroupoid determines a unique (left) ideal topology, a whole family $\mathcal{H}$ of halfgroupoids with the same elements, but different operations, may have identical (left) ideal topologies:

Example 2.1.2 Consider the following halfgroupoids:

	a	b	c	d
a	-	c	c	d
b	c	b	-	-
c	-	a	c	a
d	b	-	-	b

$\mathcal{I} = \{\phi, H, \{c\}, \{a,b,c\}\}$

	a	b	c	d
a	b	-	c	a
b	-	a	c	d
c	b	c	-	-
d	b	c	-	c

$\mathcal{I} = \{\phi, H, \{c\}, \{a,b,c\}\}$

It would of course be useful to know exactly which halfgroupoids have a given (left) ideal topology, but it appears from the above examples that there is no simple way of obtaining this information or of determining such things as the maximal and minimal halfgroupoids associated with a particular topology. It remains an open question whether these problems might be solved by defining an ordering on the set H, perhaps with regard to the (left) ideal topology.

2.2 CONTINUITY OF OPERATION

In the general theory of topological groups, it is required that the group operation be continuous under a given topology. In this section we present a sufficient condition for the operation in a halfgroupoid to be continuous with respect to the (left) ideal topology.

Definition 2.2.1 The binary operation $\circ$ in a halfgroupoid H is said to be continuous under a topology $\mathcal{T}$ if, whenever $a \circ b = c$ in H, for each $O_c \in \mathcal{T}$, there exist O_a, $O_b \in \mathcal{T}$, such that $O_a \circ O_b \subseteq O_c$.

Theorem 2.2.1 If, for every composite element c in a halfgroupoid H, every (right) factor b of c is such that $b \in \bigcap\{I_c\}$, then the operation in H is continuous under the (left) ideal topology $\mathcal{I}$.

Proof: Consider any $a,b,c \in H$ for which $a \circ b = c$, and let $I_c \in \mathcal{I}$ be an arbitrary neighborhood of c. If $b = c$, then

$I_a \circ I_c \subseteq I_c$ for any $I_a \in \mathcal{I}$, and the continuity condition is satisfied. If $b \neq c$, then c is composite and, by hypothesis, $b \in \bigcap\{I_c\}$; then $b \in (I_b \cap I_c)$, where $I_b \in \mathcal{I}$ is an arbitrary neighborhood of b. Let $(I_b \cap I_c) = I_b'$. Then $I_a \circ I_b' \subseteq I_b' \subseteq I_c$, and again the continuity condition is satisfied.

Clearly Theorem 2.2.1 gives a very strong sufficient condition for continuity of operation in a halfgroupoid, and this condition is certainly not a necessary one, as shown by the following example:

Example 2.2.1 Given the halfgroupoid

	a	b	c	d
a	b	c	b	a
b	–	–	b	c
c	c	b	–	b
d	a	b	c	–

$$\mathcal{I} = \{\phi, H, \{b,c\}, \{a,b,c\}\},$$

it can be shown that the operation in H is continuous with respect to the (left) ideal topology; however a/b, but $a \notin \bigcap\{I_b\}$ because $a \notin \{b,c\}$.

It is still an open question whether the hypothesis of Theorem 2.2.1 can be weakened in such a way as to constitute both a necessary and sufficient condition for continuity of operation in a halfgroupoid with respect to the (left) ideal topology.

2.3 SEPARATION AXIOMS

A general topological space may have various special properties, among which the separation axioms are the simplest. In this section we investigate these axioms in terms of the (left) ideal topology on a halfgroupoid.

Definition 2.3.1 A topological space $(S,\mathcal{T})$ is said to satisfy axiom T_0 if, given two different elements of S, there is an open set in $\mathcal{T}$ containing one and not the other.

Lemma 2.3.1 If two elements $a,b \in H$ are such that a/b, then $b \in \bigcap\{I_a\}$ for $I_a \in \mathcal{I}$.

Proof: Consider an arbitrary $I_a \in \mathcal{I}$. Since a/b, there exists $x \in H$ such that $x \circ a = b$. Hence $b \in (H \circ I_a) \subseteq I_a$, or $b \in I_a$. Thus, because I_a is arbitrary, $b \in \bigcap\{I_a\}$.

Theorem 2.3.1 The (left) ideal topology $\mathcal{I}$ on a halfgroupoid H satisfies axiom T_0 if, and only if, for every (right) nonzero $a \in H$, there exists an $I_a \in \mathcal{I}$ such that for all $b \in I_a$, $b \neq a$, b is not a (right) factor of a.

Proof: IF Let $a,b \in H$, where $a \neq b$. If a is a (right) zero element then $H \circ \{a\} \subseteq \{a\}$ and $\{a\}$ is a point ideal, so axiom T_0 is satisfied. If a is not a (right) zero element, by hypothesis there exists some $I_a \in \mathcal{I}$ containing no (right) factor of a. If $b \notin I_a$, axiom T_0 is satisfied. If $b \in I_a$, then $I_a \cap \{\tilde{a}\}$ contains b, but not a, and is an open set in $\mathcal{I}$ because $H \circ (I_a \cap \{\tilde{a}\}) \subseteq (H \circ I_a) \subseteq (I_a \cap \{\tilde{a}\})$, since

$(H \circ I_a) \subsetneq I_a$ and $(H \circ I_a) \subsetneq \{\tilde{a}\}$. Hence, again, axiom T_0 is satisfied.

ONLY IF Assume there exists a (right) nonzero $a \in H$ such that each $I_a \in \mathcal{I}$ contains a variable $x \in H$, $x \neq a$, where x is a (right) factor of a. Then $\bigcap\{I_a\}$, itself a neighborhood of a, contains some $b \in H$, $b \neq a$, such that b/a. But then every I_a contains b, and, by Lemma 2.3.1, $a \in \bigcap\{I_b\}$ so every I_b contains a, hence $(H,\mathcal{I})$ does not satisfy axiom T_0.

Definition 2.3.2 A topological space $(S,\mathcal{T})$ is said to satisfy axiom T_1 if, given two different elements $a,b \in S$, there exists one open set $O_a \in \mathcal{T}$ not containing b, and another open set $O_b \in \mathcal{T}$ not containing a.

Theorem 2.3.2 The (left) ideal topology $\mathcal{I}$ on a halfgroupoid H satisfies axiom T_1 if, and only if, every element of H is a point ideal.

Proof: IF Obvious.

ONLY IF Assume there exists an element $a \in H$ which is not a point ideal. Then $\bigcap\{I_a\} \neq \{a\}$ and there must exist some $b \in H$, $b \neq a$, such that $b \in \bigcap\{I_a\}$. But then every I_a contains b, contradicting the fact that $(H,\mathcal{I})$ satisfies axiom T_1.

Corollary. The topological space $(H,\mathcal{I})$ satisfies axiom T_1 if, and only if, every element of H is a (right) zero element.

Proof: IF Since a (right) zero element is a point ideal, $(H,\mathcal{I})$ satisfies axiom T_1 by Theorem 2.3.2.

ONLY IF Suppose there exists $a \in H$ such that for some $x \in H$, $x \circ a = b$, where $b \neq a$. Then a/b and, by Lemma 2.3.1, $b \in \bigcap\{I_a\}$, so $(H,\mathcal{I})$ does not satisfy axiom T_1.

We see from Theorem 2.3.2 that the conditions under which the (left) ideal topology on a halfgroupoid satisfies axiom T_1 completely characterizes $(H,\mathcal{I})$. In fact the only possible variation in the structure of the halfgroupoid is in the domain of definition of the operation, as shown by the above Corollary. Thus, although there are several other standard topological separation axioms, all of them imply T_1, so our investigation of these axioms in relation to the (left) ideal topology is now complete.

The following simple examples help to illustrate the variation in the operation on a halfgroupoid when different separation axioms are satisfied by the (left) ideal topology:

Example 2.3.1 In the halfgroupoid

	a	b	c
a	a	b	c
b	a	c	b
c	a	b	c

$$\mathcal{I} = \{\phi, H, \{a\}, \{b,c\}\}$$

we see that c is a (right) nonzero element, but every $I_c \in \mathcal{I}$ contains b and b/c; thus, by Theorem 2.3.1, $\mathcal{I}$ does not even satisfy axiom T_0.

<u>Example 2.3.2</u> In the halfgroupoid

	a	b	c
a	a	b	c
b	a	b	b
c	a	b	c

$$\mathcal{I} = \{\phi, H, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$$

we see that <u>c</u> is a (right) nonzero element, but $I_c = \{b,c\}$ contains no (right) factor of <u>c</u>; thus, by Theorem 2.3.1, $\mathcal{I}$ satisfies axiom T_0, but not T_1, by Theorem 2.3.2.

<u>Example 2.3.3</u> In the halfgroupoid

	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

$$\mathcal{I} = \{\phi, H, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$$

we see that every element is a point ideal and, by Theorem 2.3.2, $\mathcal{I}$ satisfies axiom T_1.

<u>2.4 CONNECTEDNESS</u>

In this section we discuss the conditions under which the (left) ideal topology on a halfgroupoid is connected.

<u>Definition 2.4.1</u> A topology $\mathcal{T}$ on a set S is <u>connected</u> if S is not the union of two nonempty, disjoint, open sets in $\mathcal{T}$.

Theorem 2.4.1 The (left) ideal topology $\mathcal{I}$ on a halfgroupoid H is connected if, and only if, there is no $I \in \mathcal{I}$ such that $\tilde{I} \in \mathcal{I}$.

Proof: This theorem is essentially a restatement of the definition of topological connectedness in terms of the (left) ideal topology on a halfgroupoid, but Lemma 2.4.1, below, shows under what conditions this topology actually is connected.

Lemma 2.4.1 Given any nonempty (left) ideal I in a halfgroupoid H, $\tilde{I}$ is also a (left) ideal in H if, and only if, $\tilde{I}$ is a subhalfgroupoid and $(I \circ \tilde{I}) \cap I$ is empty.

Proof: IF Since $(H \circ \tilde{I}) = (I \cup \tilde{I}) \circ \tilde{I} = (I \circ \tilde{I}) \cup (\tilde{I} \circ \tilde{I})$, and $I \circ \tilde{I} \subseteq \tilde{I}$, by hypothesis, and $\tilde{I} \circ \tilde{I} \subseteq \tilde{I}$, because $\tilde{I}$ is a subhalfgroupoid, then $H \circ \tilde{I} \subseteq \tilde{I}$ and $\tilde{I}$ is a (left) ideal in H.

ONLY IF If $\tilde{I}$ is not a subhalfgroupoid, then $\tilde{I} \circ \tilde{I} \not\subseteq \tilde{I}$ and $\tilde{I}$ is thus not a (left) ideal. Similarly, if $(I \circ \tilde{I}) \cap I$ is nonempty, then $I \circ \tilde{I} \not\subseteq \tilde{I}$ and again $\tilde{I}$ cannot be a (left) ideal.

CHAPTER THREE

APPLICATIONS

Because of the very general nature of the algebraic system which we have been discussing, and because, as the results in the preceding chapter indicate, there are some rather simple relationships between this algebraic system and a different type of mathematical system, a topology, there appear to be many opportunities for utilizing and extending the results which we have obtained herein.

In this chapter we introduce in some detail a particular extension of our connection between algebra and topology to include a third mathematical structure, the directed graph. We then discuss briefly some already known results from work being done on a problem related to our own. Finally, we present a short general survey of some possibilities for applying the notion of an ideal topology on a halfgroupoid to various other types of mathematical problems.

3.1 THE DIGRAPH TOPOLOGY

First we must state some basic definitions from the theory of graphs. For further explanation of notation related to graph theory and a more complete analysis of graphs, directed graphs, and the digraph topology, we refer to Ahlborn [1], Berge [2], Bhargava [3], [4], and Bhargava and Ahlborn [5].

<u>Definition 3.1.1</u> A <u>binary relation</u>, denoted by $\mathcal{R}$, defined on a set S, is a set E of ordered pairs, where $\phi \subsetneq E \subsetneq (S \times S)$. This set E may or may not have various special properties.

If $(a,b) \in E$, we say <u>a</u> is related to <u>b</u>, or $a \mathcal{R} b = 1$. If $(a,b) \in \tilde{E}$, then <u>a</u> is not related to <u>b</u>, or $a \mathcal{R} b = 0$.

We see that by extending the concept of a binary relation, a set of ordered pairs, as defined above, to a ternary relation, a set of ordered triples, we find that a binary operation is exactly analogous to a ternary relation, and hence now make the following alternate definition of an operation (see Definition 1.2.2):

<u>Definition 3.1.2</u> A <u>binary operation</u> defined for a set S is a set F of ordered triples, where $\phi \subsetneq F \subsetneq (S \times S \times S)$.

We note that in terms of our original definition $F = \{((a,b),c)\colon a \circ b = c \text{ in } H\}$.

<u>Definition 3.1.3</u> A <u>digraph</u> Γ (directed graph) consists of a set of elements A and a binary relation $\mathcal{R}$ defined on the set.

The digraph $\Gamma(A,E)$ is represented graphically by a set of points, or <u>vertices</u>, $A = \{a,b,c,\ldots\}$, and a set of directed <u>edges</u>, $E = \{(a,b);\ \ a,b \in A,\ a \mathcal{R} b = 1\}$, joining certain pairs of these vertices.

Definition 3.1.4 The digraph topology τ on $\Gamma(A,E)$ is the family of all subsets $U \subseteq A$ such that $(\tilde{U} \times U) \cap E = \phi$. That is, the set $U \subseteq A$ is open under τ if there are no edges in E originating in $\tilde{U}$ which terminate in U.

It has been shown (Ahlborn [1], Bhargava and Ahlborn [5]) that the family of all such subsets $U \subseteq A$ does indeed constitute a topology on the digraph $\Gamma(A,E)$. Furthermore, the mapping of the set of all possible digraphs onto the set of all digraph topologies is a many-to-one correspondence, as is the mapping of the set of all halfgroupoids onto the set of all (left) ideal topologies (see section 2.1).

We now construct mappings between the class $\mathcal{H}$ of all halfgroupoids and the class $\mathcal{D}$ of all digraphs such that under these mappings the set of all halfgroupoids $H \in \mathcal{H}$ having a particular (left) ideal topology $\mathcal{I}$ corresponds to the set of all digraphs $\Gamma \in \mathcal{D}$ having the digraph topology τ where $\mathcal{I}$ and τ are identical topologies.

In order to facilitate notation and to emphasize the similarities between halfgroupoids and digraphs, in the following discussion we will denote a halfgroupoid as (H,F), where H, as usual, is the set of elements, and F is the binary operation for H, as in Definition 3.1.2.

Construction

PART I. Let $\mathcal{H}$ be the family of all halfgroupoids and let $\mathcal{D}$ be the family of all digraphs.

Let ψ be a mapping of the set $\mathcal{H}$ into the set $\mathcal{D}$ such that $\psi(H,F) = \Gamma(H,E)$, where $E = \{(c,b)\colon ((x,b),c) \in F\}$.

Let ψ^{-1} be a mapping of the set $\mathcal{D}$ into the set $\mathcal{H}$ such that $\psi^{-1}[\Gamma(H,E)] = (H,F)$, where $F = \{((c,b),c)\colon (c,b) \in E\}$.

PART II Let H be a set of elements of arbitrary order n, and let $\mathcal{T}$ be an arbitrary topology on the set H.

Let $\mathcal{H}_0 = \{(H,F_i)\colon\ i = 1,2,\ldots\}$, where the (left) ideal topology on (H,F_i) is $\mathcal{T}$.

Let $\mathcal{D}_0 = \{\Gamma(H,E_i)\colon\ i = 1,2,\ldots\}$, where the digraph topology on $\Gamma(H,E_i)$ is $\mathcal{T}$.

In the above construction we see that the mapping ψ is a many-to-one correspondence of halfgroupoids to digraphs, and that ψ^{-1}, while not an exact inverse of ψ, has been modified only slightly in order to make it also a many-to-one (actually a one-to-one) mapping of digraphs onto halfgroupoids. The following theorem shows that these mappings ψ and ψ^{-1} establish a topological correspondence between halfgroupoids and digraphs in such a manner that the (left) ideal topology is exactly the same as the digraph topology.

Theorem 3.1.1 Given the mappings $\psi\colon \mathcal{H} \to \mathcal{D}$ and $\psi^{-1}\colon \mathcal{D} \to \mathcal{H}$ as in Part I (above), and given the sets $\mathcal{H}_0 \subseteq \mathcal{H}$ and $\mathcal{D}_0 \subseteq \mathcal{D}$ as in Part II, then (a) $\psi(\mathcal{H}_0) = \mathcal{D}_0$ and (b) $\psi^{-1}(\mathcal{D}_0) \subseteq \mathcal{H}_0$.

Proof: (a) Let $(H,F_j) \in \mathcal{H}_0$ be an arbitrary halfgroupoid having the (left) ideal topology $\mathcal{T}$, and let $\psi(H,F_j) = \Gamma(H,E_j)$. Let τ be the digraph topology on $\Gamma(H,E_j)$. We have to show

$\tau = \mathcal{T}$. Let $O \in \tau$, but assume $O \notin \mathcal{T}$. Then $H \circ O \subsetneq O$ and there must exist $b,c \in H$, $b \in O$, $c \in \tilde{O}$, such that $((x,b),c) \in F_j$. But then $(c,b) \in E_j$, so $(c,b) \in [(\tilde{O} \times O) \cap E_j]$, contradicting the fact that $O \in \tau$. Therefore, necessarily $O \in \mathcal{T}$ and, because O was chosen arbitrarily, $\tau \subseteq \mathcal{T}$. Conversely, let $O \in \mathcal{T}$, but assume $O \notin \tau$. Then $(\tilde{O} \times O) \cap E_j \neq \phi$ and there exist $b,c \in H$, $b \in O$, $c \in \tilde{O}$, such that $(c,b) \in E_j$. However, this implies there is at least one $x \in H$ such that $((x,b),c) \in F_j$, and this in turn implies $H \circ O \subsetneq O$, contradicting the fact that $O \in \mathcal{T}$. Therefore, necessarily $O \in \tau$ and, again because O was chosen arbitrarily, $\mathcal{T} \subseteq \tau$. Finally, since $\tau \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq \tau$, $\tau = \mathcal{T}$ and $\Gamma(H,E_j) \in \mathcal{D}_0$, showing that $\psi(\mathcal{H}_0) \subseteq \mathcal{D}_0$.

To show that $\psi(\mathcal{H}_0) = \mathcal{D}_0$ we first prove part (b).

(b) Let $\Gamma(H,E_k) \in \mathcal{D}_0$ be an arbitrary digraph having the digraph topology $\mathcal{T}$, and let $\psi^{-1}[\Gamma(H,E_k)] = (H,F_k)$. Let $\mathcal{I}$ be the (left) ideal topology on (H,F_k). We have to show $\mathcal{I} = \mathcal{T}$. Let $O \in \mathcal{I}$, but assume $O \notin \mathcal{T}$. Then $(\tilde{O} \times O) \cap E_k \neq \phi$ and there exist $b,c \in H$, $b \in O$, $c \in \tilde{O}$, such that $(c,b) \in E_k$. But this means $((c,b),c) \in F_k$ and $H \circ O \not\subseteq O$, contradicting the fact that $O \in \mathcal{I}$. Hence necessarily $O \in \mathcal{T}$ and, because O is arbitrary, $\mathcal{I} \subseteq \mathcal{T}$. Conversely, let $O \in \mathcal{T}$ and assume $O \notin \mathcal{I}$. Then $H \circ O \not\subseteq O$ and there exist $b,c \in H$, $b \in O$, $c \in \tilde{O}$, such that $((c,b),c) \in F_k$. (We note that by the construction of (H,F_k) automatically the general element $((x,b),c) \notin F_k$.) But $((c,b),c) \in F_k$ implies $(c,b) \in E_k$, so $(c,b) \in [(\tilde{O} \times O) \cap E_k]$, contradicting the fact that $O \in \mathcal{T}$. Hence necessarily $O \in \mathcal{I}$ and, because O is arbitrary, $\mathcal{T} \subseteq \mathcal{I}$. Finally, because $\mathcal{I} \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{I}$, $\mathcal{I} = \mathcal{T}$ and $(H,F_k) \in \mathcal{H}_0$, showing that $\psi^{-1}(\mathcal{D}_0) \subseteq \mathcal{H}_0$. It is now clear that $\psi(\mathcal{H}_0) = \mathcal{D}_0$ since every $\Gamma(H,E_i) \in \mathcal{D}_0$ has the inverse image in $\mathcal{H}_0$ given by the definition of ψ^{-1} in Part I of our construction.

Summarizing briefly, in this section we began by redefining a binary operation as a ternary relation, so as to make more apparent a rather natural connection between halfgroupoids and digraphs. We then presented the standard definition of the digraph topology and showed that our (left) ideal topology is identical with the digraph topology under appropriate mappings between halfgroupoids and digraphs, thereby establishing connections among the three different mathematical fields of algebra, topology, and graph theory.

3.2 A CONVERSE PROBLEM

Throughout this thesis our approach to the study of the relationships between an algebraic system and a topology has been generally that of determining the structure of a halfgroupoid, given the properties of the corresponding (left) ideal topology. A considerable amount of research has been done by Hanson (see [9]) on a somewhat converse problem, that of determining topological structures on a given algebraic system. He limits his consideration almost entirely to groupoids and to those topologies under which the groupoid operation is continuous (or compatible). He begins with an investigation of topologically trivial systems (those algebraic systems for which the only permissible topologies are the discrete and indiscrete), and then continues, analyzing systems possessing various types of more specialized topologies. He also studies the concept of generalized ideal (a subset of a groupoid G whose complement is a basis of singleton sets for an admissible topology on G). In general, Hanson's work is of a complementary nature to ours in the relating of algebra and topology, and though there are probably meaningful ways of correlating these two approaches, such a problem is beyond the scope of this thesis.

3.3 CONCLUDING REMARKS

Whenever a significant relationship between apparently different, but abstractly similar, mathematical systems is discovered, the possibilities for applying the results are automatically multiplied. Information known in terms of one system can be more readily interpreted in terms of the other, and solutions to problems in one field may often produce corresponding solutions to similar problems in the related field.

By first establishing a simple connection between algebra and topology and then, in turn, relating these to directed graphs, we have perhaps opened one way for problems in such diverse fields as number theory, probability, statistics, group theory, and point-set topology, to be considered from several different points of view. In this way it is hoped that new insight might be gained into the solutions particularly of various counting and maximal-minimal problems, as well as the answers to many other questions pertinent to these fields of mathematics.

BIBLIOGRAPHY

1. AHLBORN, THOMAS J. "On Directed Graphs and Related Topological Spaces." Unpublished Master's thesis, Kent State University, 1964.

2. BERGE, CLAUDE. The Theory of Graphs and Its Applications. Translated by A. Doig. New York: Wiley and Sons, Inc., 1962.

3. BHARGAVA, T. N. "A Stochastic Model for Time Changes in a Binary Dyadic Relation, with Application to Group Dynamics." Unpublished Ph.D. dissertation, Michigan State University, 1962.

4. ______. "Time Changes in a Digraph," Journal of Applied Probability, 2 (June, 1965).

5. BHARGAVA, T. N., and AHLBORN, T. J. "On Digraph Topology." (Submitted for publication to Mathematische Annalen).

6. BHARGAVA, T. N., and OHM, S. E. "Some Results on Algebraic Structures and the Digraph Topology," Abstract No. 622-54, Notices of the American Mathematical Society, 12 (April, 1965), 341.

7. BRUCK, RICHARD H. A Survey of Binary Systems. Vol. 20: Ergebnisse der Math. Neue folge. Berlin: Springer-Verlag, 1958.

8. DOYLE, P. H., and WARNE, R. J. "Some Properties of Groupoids," The American Mathematical Monthly, 70 (December, 1963), 1051-1057.

9. HANSON, JOHN R. Unpublished Ph.D. dissertation on topologically trivial algebraic systems, Virginia Polytechnic Institute, 1965.

10. KELLEY, JOHN L. General Topology. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1955.

APPENDIX

In addition to the approach followed in this thesis for linking first algebra and topology, and then algebra and graphs, through a topological correspondence, there are several other ways of establishing connections among these three mathematical systems, and here we will very briefly outline two other approaches along which some research has already been done and from which useful results have been obtained (see Bhargava and Ohm [6]). For definitions of special terms related to graphs, we refer particularly to Berge [2].

First, there are many different mappings which may be defined between the family of all halfgroupoids and the family of all digraphs, in addition to the ones already given in section 3.1. For example, the following three have been studied to some extent and appear to show promise of leading to further useful results:

i) $\psi_1(H,F) = \Gamma(H,E)$, where $(a,b) \in E$ if $\big((a,b),x\big) \in F$;

ii) $\psi_2(H,F) = \Gamma(H,E)$, where there is a path from $\underline{a}$ to $\underline{b}$ if $\big((a,b),x\big) \in F$;

iii) $\psi_3(H,F) = \Gamma(H,E)$, where $(a,c) \in E$ if $\big((a,x),c\big) \in F$.

It was found that of these three mappings, ψ_3 defines the most generally interesting relationship between halfgroupoids and digraphs: for example, a cyclic groupoid yields a graph possessing a Hamiltonian line whose length is the order of the generating element of the groupoid; prime elements map into unaccessible points, and idempotents into loops; an antihalfgroupoid is related to the kernel of a graph, and associativity or commutativity of the halfgroupoid operation to the valency or density of vertices in the graph.

The second main approach investigated was that of linking algebra and topology through a two-sided ideal topology on a halfgroupoid, and theorems corresponding to those appearing in Chapter Two were first obtained using these two-sided ideals. However, it was then observed that, by considering the more general case of a one-sided ideal, a topology could still be obtained, and, furthermore, that by using a modified version of the mapping ψ_3, the topological correspondence as described in section 3.1 could be obtained.

Thus it was a combination of the two approaches outlined above that was finally followed in developing this thesis. However, it is still felt that the many various possible relationships between these different mathematical fields should be further explored, in order to obtain whatever results which may be of value when applied to various situations.